Semiclassical resolvent bounds in dimension two

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Definitons and notation

- Let $\Delta \leq 0$ be the Laplacian on $\mathbb{R}^2$.
- We consider semiclassical Schrödinger operators of the form

$$P = P_h := -h^2\Delta + V - E, \qquad E, h > 0.$$

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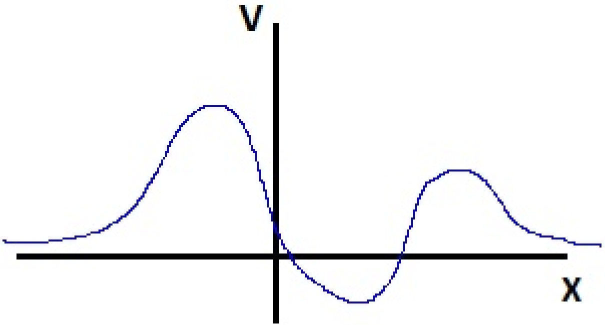
- Assume $V(x), \nabla V(x)$ belong to $L^\infty(\mathbb{R}^2)$, and that they are real-valued.

Conditions on V(x)

- Set $r = |x|$. Suppose there exists $\delta_0 > 0$ so that the following inequalities hold for almost all $x \in \mathbb{R}^2$,

$$V(x) \leq \frac{1}{2}(1+r)^{-\delta_0}, \qquad |\nabla V(x)| \leq \frac{1}{2}(1+r)^{-1-\delta_0}.$$

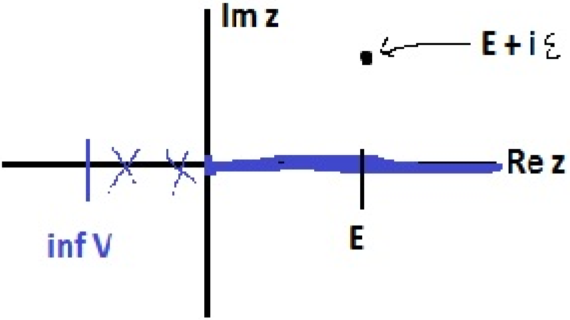


Spectrum of $V(x)$

- The Kato Rellich Theorem shows that, for $\varepsilon > 0$, the resolvent

$$(P - i\varepsilon)^{-1} = (-h^2\Delta + V - E - i\varepsilon)^{-1}$$

is a bounded linear operator $L^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$.



Resolvent estimates

- We establish bounds between the weighted spaces $L^2_s \to L^2_{-s}$.

Theorem ([Sh16])

For any $s > 1/2$ there are $C, R, h_0 > 0$ such that

$$\left\|(1+r)^{-s}(P - i\varepsilon)^{-1}(1+r)^{-s}\right\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq e^{\frac{C}{h}},$$

$$\left\|(1+r)^{-s}\mathbf{1}_{\geq R}(P - i\varepsilon)^{-1}\mathbf{1}_{\geq R}(1+r)^{-s}\right\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq \frac{C}{h},$$

for all $\varepsilon > 0$ and $h \in (0, h_0]$, where $\mathbf{1}_{\geq R}$ is the characteristic function of $\{x \in \mathbb{R}^2 \colon |x| \geq R\}$.

Previous results

- Burq [Bu98, Bu02] proved exponential resolvent estimates in the context of an obstacle problem on an exterior domain $\mathbb{R}^n \setminus \overline{\mathcal{O}}$, $n \geq 1$.
- Cardoso and Vodev [CaVo02] extended Burq's results to a wide class of infinite volume Riemannian manifolds.
- Rodnianski and Tao [RT15] obtained similar resolvent estimates for Schrödinger operators on asymptotically conic manifolds.

Previous results

- Vodev [Vod14] worked in $\mathbb{R}^n$, $n \geq 3$ and replaced V by $h^\nu V \in L^\infty(\mathbb{R}^n)$, $\nu > 0$.
- Datchev [Da14] gave a proof of the resolvent bounds in dimension $n \geq 3$. He only required a decay condition on $\partial_r V$.
- The novel aspect of the Theorem is that we have proved the bounds in dimension two while only requiring low regularity and mild decay on V and ∇V.

Previous results

- The h dependence in

$$\left\|(1+r)^{-s}(P-i\varepsilon)^{-1}(1+r)^{-s}\right\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \geq e^{\frac{C}{h}}$$

is well-known to be optimal in general. Moreover, in [DDZ15], Datchev, Dyatlov, and Zworski established the lower bound

$$\left\|(1+r)^{-s}\mathbf{1}_{\geq R}(P-i\varepsilon)^{-1}(1+r)^{-s}\right\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \geq e^{\frac{1}{Kh}},$$

for a suitable class of V.

Previous results

- Stronger resolvent bounds are known if V is more regular, and additional assumptions are made about solutions to the Hamiltonian system.

$$\dot{x} = 2\xi, \qquad \dot{\xi} = -\nabla_x V.$$

For example, if V is *nontrapping* at the energy E, then it is known that the resolvent bound can be improved to

$$\left\|(1+r)^{-s}(P - i\varepsilon)^{-1}(1+r)^{-s}\right\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq \frac{C}{h}.$$

For more about resolvent bounds under various dynamical assumptions, see [Wun12], as well as chapter 6 from [DyZw], and the references therein.

Carleman estimate

- Key to proof of Theorem is to produce a global Carleman estimate that holds for any $v \in C_0^\infty(\mathbb{R}^2), \varepsilon \geq 0$, and $h \in (0, h_0]$. We establish

$$\left\|(1+r)^{-s} e^{\varphi(r)/h} v\right\|_{L^2}^2 \leq \frac{C}{h^2}\left\|(1+r)^{s} e^{\varphi(r)/h}(P - i\varepsilon) v\right\|_{L^2}^2 + \frac{C\varepsilon}{h}\|e^{\varphi(r)/h} v\|_{L^2}^2.$$

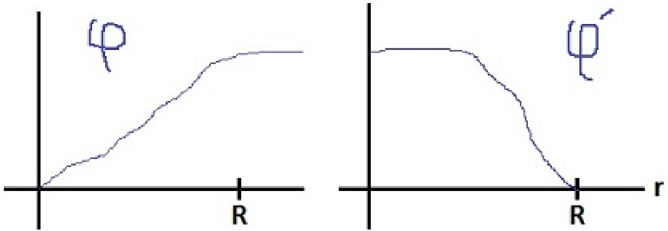
- The constants $R, C > 0$ are independent of v and ε.
- The Theorem then follows by a density argument due to Datchev [Da14].

The Carleman weight $\varphi(r)$

- [DdeH16] $\varphi \in C^2([0,\infty))$ is the unique solution to

$$(\varphi'(r))^2 - h\varphi''(r) = \psi(r).$$

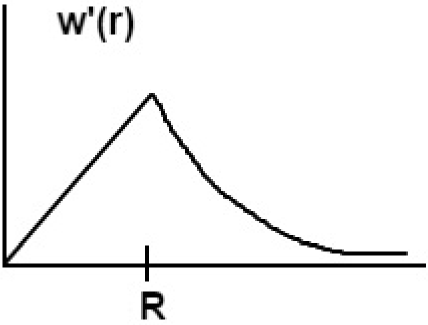
- [Da14] ψ is a continuous function that obeys a certain inequality with E, V, and $\partial_r V$.
- $\varphi' \geq 0$, supp$\varphi' \subseteq [0, R]$, and the support is independent of h.



The function $w(r)$

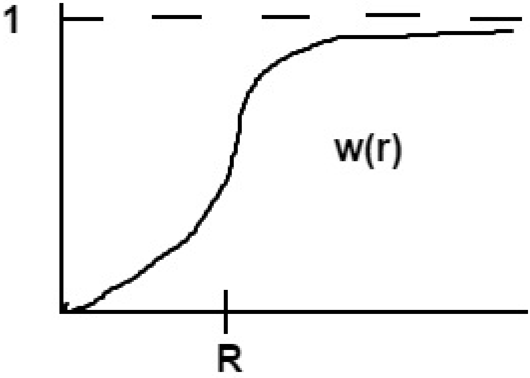
- Set $\delta := 2s - 1$ (WLOG can take $\delta < \delta_0$).

$$w'(r) := 2c_0 r \mathbf{1}_{\leq R} + (1 + r)^{-1-\delta} \mathbf{1}_{\geq R}.$$



The function $w(r)$

$$w = w_\delta(r) := \begin{cases} c_0 r^2, & r \le R, \\ 1 - (1+r)^{-\delta}, & r > R, \end{cases}$$



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- Define $V_\varphi := V - (\varphi')^2 + h\varphi'' - h^2/4r^2$. Set

$$\begin{aligned} P_\varphi &:= e^{\frac{\varphi}{h}} r^{\frac{1}{2}} (P - i\varepsilon) r^{-\frac{1}{2}} e^{-\frac{\varphi}{h}} \\ &= -h^2\partial_r^2 + 2h\varphi'\partial_r - \frac{h^2}{r^2}\Delta_{\mathbb{S}^1} + V_\varphi - E - i\varepsilon. \end{aligned}$$

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- In general, if $n \geq 2$ is the dimension, $r^{\pm 1/2} \to r^{\pm(n-1)/2}$, so $-h^2/4r^2$ becomes $(n-1)(n-3)h^2/4r^2$.

Carleman estimate

- Goal: for $u \in e^{\frac{\varphi}{h}} r^{\frac{1}{2}} C_0^\infty(\mathbb{R}^2)$, make integral estimates with aid of the functional

$$F(r) := \|h\partial_r u(r,\theta)\|_S^2 - \langle(-h^2 r^{-2}\Delta_{\mathbb{S}^1} + V_\varphi(r,\theta) - E)u(r,\theta), u(r,\theta)\rangle_S.$$

- We can show $\int_0^\infty (wF)' \leq 0$, and from this we get

$$\int_{r,\theta} (w(E - V_\varphi))'|u|^2 \leq \frac{1}{h^2} \int_{r,\theta} \frac{w^2}{w'} |P_\varphi u|^2 + 2\epsilon \int_{r,\theta} w|uu'|$$

with respect to the measure $drd\theta$.

Preliminary inequality

Lemma ([Sh16])

If $\delta > 0$ *is small enough, and*

$$V \leq (1+r)^{-\delta_0}, \qquad |\partial_r V| \leq (1+r)^{-1-\delta_0},$$

then there exists $h_0 > 0$ *so that*

$$\partial_r \left(w(r)(E - V_\varphi(r,\theta)) \right) \geq \frac{E}{4} w'(r),$$

for almost all $(r, \theta) \in \mathbb{R}^2 \setminus \{0\}$ *and any* $h \in (0, h_0]$.

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- Expanding $\partial_r(w(E - V_\varphi))$ and making use of

$$(\varphi'(r))^2 - h\varphi''(r) = \psi(r),$$

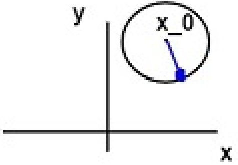
we get,

$$\partial_r(w(E-V_\varphi)) = (E-V+\psi)w' + (-\partial_r V + \psi')w + \frac{h^2}{4r^2}\left(w' - \frac{2w}{r}\right).$$

Carleman estimate

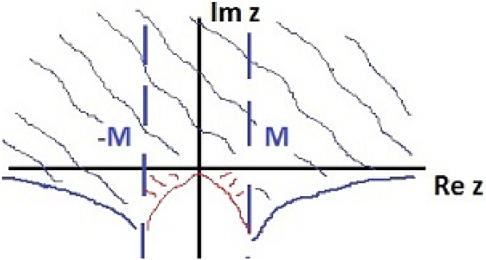
$$\left\| c r^{\frac{1}{2}} \mathbf{1}_{\leq R} e^{\varphi/h} v \right\|_{L^2}^2 + \left\| (1+r)^{-s} \mathbf{1}_{\geq R} e^{\varphi/h} v \right\|_{L^2}^2 \leq \frac{C}{h^2} \left\| (1+r)^s e^{\varphi/h} (P - i\varepsilon) v \right\|_{L^2}^2 + \frac{C\varepsilon}{h} \| e^{\varphi/h} v \|_{L^2}^2 .$$

- Estimate is weak near the origin. Can shift coordinates to a new origin x_0. Shifted potential still satisfies needed bounds. Then add estimates together.



Application: resonance free regions

- Burq established resonance free regions of the form $\{z \in \mathbb{C} : -\operatorname{Im} z < \varepsilon_0 e^{-C|\operatorname{Re} z}, |\operatorname{Re} z| \geq M\}$.



- Resonance free regions allow contour deformation and nice integral estimates.

Application: wave decay

- Recall energy conservation for solutions to the wave equation $(\partial_t^2 - \Delta)u(x,t) = 0$ in $\mathbb{R}^n$:

$$\frac{1}{2}\int_{\mathbb{R}^n} |\partial_t u|^2 + |\nabla u|^2 dx = \text{constant}.$$

Theorem ([Bu98, Bu02])

For any $R_1, R_2 > 0$, there exists $C > 0$ such that for any initial conditions u_0, u_1 with support in $B(0, R_1)$, the solution u obeys

$$E_{R_2}(u)(t) := \int_{B(0,R_2)\cap(\mathbb{R}^n \setminus \overline{\mathcal{O}})} |\nabla u|^2 + c^{-2}|\partial_t u|^2 dx \leq \frac{C}{\log(2+t)}(\|u_0\|^2_{H_0^1} + \|u_1\|^2_{L^2}).$$

Application: wave decay

- We aim to establish logarithmic local energy decay for wave equations with rough wavespeed: we assume

$$c, c^{-1} \in L^\infty(\mathbb{R}^n), 1 - c \in L^\infty_{\text{comp}}.$$

- Difficulty in the L^∞ case is the presence of a resonance at $z = 0$.
- Jensen and Nenciu [JeNe] give explicit expansions in low dimensions for the integral kernel of the resolvent near zero. In $\mathbb{R}^2$, the singularity is $\log z$.

Application: resonance counting

- In the recent paper [Chr15], Christiansen used the resolvent bound near infinity to find a lower bound on the resonance counting function on even-dimenisonal Riemannian manifolds that are flat near infinity and contain a compact perturbation.

$$\left\|(1+r)^{-s}\mathbf{1}_{\geq R}(P-i\varepsilon)^{-1}\mathbf{1}_{\geq R}(1+r)^{-s}\right\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leq \frac{C}{h},$$

Future work

- Question: how do the constants in the resolvent estimates depend on the energy level E?

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