Scattering resonances with applications to wave decay

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1. Introduction

- (Show video of Tacoma bridge collapse.) "When you think of the concept of resonance, you might think of the Tacoma bridge collapse, which we just watched, or you might think of a symphony playing a strong piece of music." The type of resonance that I study is related to those things, but it comes from the quantum mechanical world (micro instead of macro). Resonance is mathematically interesting in its own right, but it also allows allows us to address the physical question: How does a wave evolve over time?
- If anyone has a question as we go along, please ask!
- Also, I hope that this talk serves as something of an advertisement for those of you who are interested in functional analysis and PDE. Please don't hesistate to come to talk to me later if you would like some references on this subject.

2. Mathematical preliminaries

First goal: Understand some of the basic mathematical tools used to study resonance. See resonance in the simplest case of the one dimensional free resolvent.

The basic setting in which we study resonances is the Hilbert space $L^2(\mathbb{R}^n)$.

(1) We consider the vector space of Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\int_{\mathbb{R}^n} |f|^2 dx < \infty$. The inner product on $L^2(\mathbb{R}^n)$ is given by $\langle f, g\rangle_{L^2(\mathbb{R}^n)} := \int_{\mathbb{R}^n} f\bar{g} dx$. The norm is $\|f\|_{L^2(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f|^2 dx\right)^{\frac{1}{2}}$.

(2) If we want to understand how a system is changing, we need to be able to take derivatives. But we quickly run into a problem with this because L^2 functions are not necessarily differentiable, let alone pointwise defined. Luckily, *distribution theory* has been developed to deal with this inconvenience. The theory was used informally by physicists for centuries. It was made rigorous by Laurent Schwartz in the 1940s. Let's look at a little bit of distribution theory.

We work in one dimension for simplicity, but this concept extends easily to $\mathbb{R}^n$. The first fact we need to have is that functions $\varphi \in C_0^\infty(\mathbb{R})$ exist (this takes some work to show!). These functions are wonderful for many reasons: they have derivatives of all orders, they are integrable, bounded, and dense in $L^2(\mathbb{R})$. Whenever we can, we want to work with functions in $C_0^\infty(\mathbb{R})$. Next, Let $f \in L^2(\mathbb{R})$. We say that $v : \mathbb{R} \to \mathbb{C}$ is the **weak n-th derivative** of f if and only if for all $\varphi \in C_0^\infty(\mathbb{R})$, we have $\int f \frac{d^n \varphi}{dx^n} dx = (-1)^n \int_{\mathbb{R}^n} v dx$. If this is the case, then v is in fact unique, and we write $v = \frac{d^n f}{dx^n}$. What's the point? Well, if function has a weak derivative, we can still integrate by parts, so long as we integrate against a test function.

The **Sobolev Space** $H^2(\mathbb{R})$ is the subspace $\{f \in L^2(\mathbb{R}) : \frac{df}{dx}, \frac{d^2 f}{dx^2}$ exist and belong to $L^2(\mathbb{R})\}$. How bad can these functions actually be? Depending on the dimension, they are not bad at all. Nonetheless, these definitions are used even in simple cases (like $n = 1$) because theory has proved to be so powerful.

(3) We also need to use the Fourier transform. For now, just assume that $\varphi(x) \in C^\infty(\mathbb{R})$ and that it decays very rapidly (faster than any inverse polynomial). We define the **Fourier transform** of φ to be

$$\mathcal{F}(\varphi)(\xi) := \int_{\mathbb{R}} e^{-ix\xi} \phi(x) dx, \qquad \xi \in \mathbb{R}.$$

Note that when we take the Fourier transform, we switch to the variable ξ. What makes the Fourier transform so useful is what it does to differentiation and multiplication by polynomials. We set $D_x = \frac{1}{i}\frac{d}{dx}$. Then we have

$$\mathcal{F}(D_x\varphi(\cdot))(\xi) = \xi\mathcal{F}(\varphi)(\xi), \qquad \mathcal{F}((-ix)\varphi(\cdot))(\xi) = \frac{d}{d\xi}\mathcal{F}(\varphi)(\xi).$$

The Fourier transform also has a very nice property with the **convolution** operation. The convolution operation is

$$\varphi * \psi(x) := \int_{\mathbb{R}} \varphi(x-y)\psi(y)dy, \qquad \varphi, \psi \in C^\infty(\mathbb{R}) \text{ rapidly decaying.}$$

And the nice the interaction with the Fourier transform is given by

$$\mathcal{F}(\varphi * \psi)(\xi) = \mathcal{F}(\varphi)(\xi)\mathcal{F}(\psi)(\xi).$$

Again, these concepts all extend to higher dimensions, but we just present the simplest case.

Another major tool used in the study of resonances is the theory of linear operators and their spectra.

(1) A linear map $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is **bounded** if and only if there exists a constant $C > 0$ such that $\|Tf\| \leq C\|f\|$ for all $f \in L^2(\mathbb{R}^n)$. This is another way to say that the function T is continuous with respect to the norm on $L^2(\mathbb{R}^n)$. The **norm** $\|T\|$ of T is the smallest such C that we can use. The set of bounded operators is denoted $\mathcal{B}(L^2(\mathbb{R}^n))$. A simple example of such a map is multiplication by a function $V(x) \in L^\infty(\mathbb{R}^n)$. The Fourier transform can also be extended from the rapidly decaying functions to become a bounded linear operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

If a map T is linear but there is no such C as described above, then we just say that T is unbounded. Usually in this case the domain $D(T)$ of T is smaller than the whole $L^2(\mathbb{R}^n)$ space. For us, the key example of an unbounded operator occurs in the context of $H^2(\mathbb{R})$. We take $T = D_x^2 = -\frac{d^2}{dx^2}$, $D(T) = H^2(\mathbb{R})$, and define $T(f) = -\frac{d^2 f}{dx^2}$.

Now we need to talk a little about *inverses* of linear operators. Let T be an unbounded operator on $L^2(\mathbb{R}^n)$ with domain $D(T)$. We say that a point $z \in \mathbb{C}$ belongs to the **resolvent set** $\rho(T)$ if and only if the operator $(T-z)^{-1} : L^2(\mathbb{R}^n) \to D(T)$ is exists and is bounded with respect to the L^2 norm. Otherwise, we say that z belongs to the **spectrum** of T, denoted $\sigma(T)$.An **eigenvalue** of T is a number $z \in \mathbb{C}$ such that there exists a nonzero $u \in D(T)$ such that $Tu = zu$. Note that if z is an eigenvalue of T, then $z \in \sigma(T)$.

One important example: the point $z = 0$ belongs to the spectrum of the Fourier transform $\mathcal{F}$, with domain taken to be all of $L^2(\mathbb{R}^n)$. That is, the Fourier transform is continuosly invertible on $L^2(\mathbb{R}^n)$.

It's an amazing (or at least really cool) fact that we can do complex analysis with Linear operators. Let's see a little bit how this works. $\Omega \subseteq \mathbb{C}$ be an open, connected set. Suppose that $A : \Omega \to \mathcal{B}(L^2(\mathbb{R}^n))$. Then $A(z)$ is **holomorphic** in Ω if and only if there exists $B : \mathbb{C} \to \mathcal{B}(L^2(\mathbb{R}^n))$ such that, for all $z_0 \in \Omega$, we have

$$\lim_{z \to z_0} \left\| \frac{A(z) - A(z_0)}{z - z_0} - B(z_0) \right\| = 0.$$

In this case, we write $\frac{dA}{dz} = B$. Example: Let T be a linear operator and suppose that $z_0 \in \rho(T)$. Then the map $z \mapsto (T-z)^{-1}$ is holomorphic in a neighborhood of z_0 with derivative equal to $-(T-z)^{-2}$.

In complex analysis, we don't just have holomorphic functions, we have meromorphic functions as well. We also have meromorphic functions when we map into spaces of linear operators.

We say that $z \mapsto A(z)$ is a **meromorphic** family of operators in Ω if and only if for any $z_0 \in \Omega$, there exist operators A_j, $1 \leq j \leq J$ of *finite rank* and a family of operators $z \mapsto A_0(z)$ holomorphic in a neighborhood of z_0 such that

$$A(z) = A_0(z) + \frac{A_1}{z - z_0} + \cdots + \frac{A_J}{(z - z_0)^J}.$$

We have a **simple** pole if $A_2 = \cdots = A_J = 0$.

Now we have all the ingredients to give the definition of resonance, and look at the simplest case. Again, we scale back to working in just one dimension.

(1) We define P_V to be the operator $P_V := D_x^2 + V(x) : H^2(\mathbb{R}) \to L^2(\mathbb{R})$, where $V(x)$ is a bounded compactly supported function (i.e., $V \in L^\infty_{\text{comp}}(\mathbb{R})$). For the moment we only consider $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda > 0$ (that is, λ in the upper half plane). We ask: does the resovlent $R_V(\lambda) := (P_V - \lambda^2)^{-1} : L^2(\mathbb{R}) \to H^2(\mathbb{R})$ exist in some part of the upper half plane? If so, can we meromoprhically continue the map $\lambda \mapsto R_V(\lambda)$ through the real axis and into the lower half of the complex plane? If we can do this, then the **resonances** are defined to be the poles of this continuation. The catch to all this: the resolvent may no longer be an operator from $L^2(\mathbb{R}) \to H^2(\mathbb{R})$. We have to adjust the domain and range spaces to get a well-defined continuation.

One might wonder why we using λ^2 now instead of z. One explanation is that the solution u to $(D_x^2 - z)u = f$ will turn out to depend on $\sqrt{z}$, which we means we must implicitly choose a branch of the square root function. We have effectively already done this by our requirement $\operatorname{Im} \lambda > 0$.

In the simplest case of $n = 1$ and $V \equiv 0$, we can come up with an explicit formula for the continuation. Point out why things go wrong when $\operatorname{Im} \lambda < 0$, and say what we do to fix that. With more general V, a continuation is still possible, but knowing exactly where the resonances are gets more difficult. But usually, it's enough to know where the resonances *aren't* as opposed to where they *are*.

3. Applications

Second goal: See how scattering resonances play a similar role to eigenvalues for problems on non compact domains. Understand wave decay in a simple case.

Consider first the wave equation on bounded domain.

$$\begin{cases} D_t^2 u - P_V u = 0, & (x,t) \in [a,b] \times (0,\infty), \\ u(x,0) = u_0(x) \in C_0^\infty([a,b]), \\ D_t u(x,0) = u_1(x) \in C_0^\infty([a,b]) \\ u(t,a) = u(t,b) = 0 \end{cases}$$

Solutions to this equation can represented as superpositions in terms of eigenvalues of the of the operator P_V. On the other hand, we can consider the wave equation in an unbounded domain.

$$\begin{cases} D_t^2 u - P_V u = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x) \in C_0^\infty(\mathbb{R}), \\ D_t u(x,0) = u_1(x) \in C_0^\infty(\mathbb{R}). \end{cases}$$

And then we have the following theorem.

Theorem. *For any $A > 0$, there are $\delta, R > 0$ such that the region $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \geq -A - \delta \log(1 + |\operatorname{Re} \lambda|), |\operatorname{Re} \lambda| \geq R\}$ is resonance free. Suppose $u(x,t)$ is the solution of the preceding equation. Then*

$$u(x,t) = \sum_{\operatorname{Im}\lambda>0} \operatorname{Res}[(iR_V(\lambda)u_1 + \lambda R_V(\lambda)u_0)e^{-i\lambda t}]$$
$$+ \sum_{0>\operatorname{Im}\lambda\geq -A-\delta\log(1+|\operatorname{Re}\lambda|)} \operatorname{Res}[(iR_V(\lambda)u_1 + \lambda R_V(\lambda))u_0 e^{-i\lambda t}] + E_A(t).$$

Furthermore, for any $K > 0$ *such that* $\operatorname{supp} u_j \subseteq [-K, K]$ *(*$j = 0, 1$*), there exist constants* $C_{K,A}$ *and* $T_{K,A}$ *such that* $\|E_A(t)\|_{H^2([-K,K])} \leq C_{K,A}e^{-tA}(\|u_0\|_{H^1} + \|u_1\|_{L^2})$ *when* $t \geq T_{K,A}$.

First, this theorem says that, up to a relatively small error, the solution u is equal to a sum of residues over resonant states. Second, the theorem says that, in compact sets, u decays exponentially fast, so long as we exclude a sum over finitely many resonant states.